Existence and uniqueness of solution for conformable sequential differential equations

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Received 19/3/2017 *Accepted* 13/5/2017

Abstract:.

In this article, we obtain sufficient conditions for existence and uniqueness of sequential nonlinear differential equations involving conformable derivatives with any order less than or equal three. We use Schauder and Banach fixed point theorems to prove the existence and uniqueness problems. Some examples are introduced to illustrate the theoretic results.

Key words: Existence and uniqueness; conformable differential equations; Fixed point theorems.

1 Introduction

The amazing results of applying the fractional order derivatives in the models of many underlying phenomena attracted the researchers to investigate in-depth work about various directions of fractional calculus (see [1]-[3] and the references cited therein). Among these investigations, the existence theory of solutions for fractional differential models has gained attentions of many authors. Most of them have focused on using Riemann-Liouville and Caputo derivatives in representing the underlying fractional differential equation (see [4]-[8]). The existence problem using other fractional derivatives has not sufficiently considered by researchers. A new fractional derivative approaches in the literature called conformable derivatives was appeared recently (see [9], [11], [12], [13]). The differential systems that would be modeled by these derivatives are considered as theoretical view since the conformable derivatives are close to classical derivatives in their properties and hence in the physical interpretations of real applications. Motivated by the literature, we consider some sequential differential equations of fractional orders less than or equal to three and we obtain the existence of solutions of such equations. The considered type is different from the conformable type studied in [16] and similar to the Caputo (classical) fractional one considered in [15].

More precisely, we consider the nonlinear conformable differential equations given by

$$\begin{cases} \boldsymbol{T}_{\alpha}^{a}x(t) = f_1\big(t, x(t)\big), 0 < \alpha \leq 1\,, \\ x(a) = x_0, \end{cases} \tag{1.1}$$

$$\begin{cases} (\boldsymbol{T}_{\alpha}^{a} + \gamma \boldsymbol{T}_{\boldsymbol{\alpha-1}}^{\boldsymbol{a}})x(t) = f_2\big(t, x(t)\big), 1 < \alpha \leq 2\,, \\ x(a) = x_0, x^{'}(a) = x_1, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \left(\boldsymbol{T}_{\boldsymbol{\alpha}}^{\boldsymbol{a}} + \lambda \boldsymbol{T}_{\boldsymbol{\alpha-1}}^{\boldsymbol{a}} + \frac{\lambda^2}{4}\boldsymbol{T}_{\boldsymbol{\alpha-2}}^{\boldsymbol{a}}\right)x(t) = f_3\big(t, x(t)\big)\,, 2 < \alpha \leq 3, \\ x(a) = x_0, x^{'}(a) = x_1, x^{''}(a) = x_2, \end{cases} \tag{1.3}$$

where $t \in J = [a, T]$, $0 \leq a < T$, $f_k: J \times \mathbb{R} \rightarrow \mathbb{R}, k = 1,2,3,$ are given continuous functions, γ and λ are real numbers, and $\boldsymbol{T}_{\boldsymbol{\alpha}}^{\boldsymbol{a}}$ are conformable fractional derivatives.

2 Linear Conformable Differential Equations

The new fractional derivative that is called conformable derivative, was introduced firstly by Khalil et al. [9], then is developed by Abdeljawad [11], then a newer definition called Katugampola derivative ([12], [13]) is appeared. The definitions have almost the same integral definition and properties, that are necessary in this study, we can say that the results of this study still true to both definition, so we call it as conformable fractional derivative.

Let $\boldsymbol{C}(J,\mathbb{R})$denotes the Banach space of all real valued continuous functions endowed with the norm $\{\|x\| = \sup\{|x(t)|: t \in J\}$. The space $\boldsymbol{C^n}(J,\mathbb{R})$is a Banach space of all *n*th differentiable real valued functions defined on J.

We firstly, present the definitions and some properties of the conformable fractional integrals and derivatives.

Definition 2.1 [11] The conformable derivative of order $\alpha \in (0,1]$ of a function f is defined by

$$f^{\alpha}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > a$. If the limit exists, then we say f is α-differentiable. If $\lim_{t\to a^+} f^{\alpha}(t)$ exists, then we define $f^{\alpha}(a) = \lim_{t\to a^+} f^{\alpha}(t)$.

Many classical properties such as the linearity, chain rule, etc. are valid by applying Definition 2.1. Abdeljawad et. al. [11] studied Definition 2.1 and introduced some basic concepts about it.

Definition 2.2 The conformable derivative of order α, denoted by $\boldsymbol{T}_{\alpha}^{a}$, of $f \in \boldsymbol{C^{n+1}}(J,\mathbb{R})$ is defined as

$$\boldsymbol{T}_{\alpha}^{a} f(t) = (t-a)^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t),$$

where $\alpha \in (n, n+1]$, and $t > a$.

The analogous definition of the integral operator corresponding to derivative operator $\boldsymbol{T}_{\alpha}^{a}$ is given by the following definition.

Definition 2.3 [11] The conformable integral of order $\alpha > 0$, denoted by $\boldsymbol{I}_{\alpha}^{a}$, of $f \in \boldsymbol{C}(J,\mathbb{R})$ is defined as

$$\boldsymbol{I}_{\alpha}^{a} f(t) = I_{a}^{n+1}\big((t-a)^{\beta-1} f(t)\big),$$

where $\beta = \alpha - n$, $\alpha \in (n, n+1]$, and $I_{a}^{n+1}$is the Riemann–Liouville integral operator given by [10]

$$I_{a}^{q} f(t) = \frac{1}{\Gamma(q)} \int_{a}^{t} (t-s)^{q-1} f(s) ds.$$

We set $\boldsymbol{I}_0^a f(t) = f(t)$. Notice that if $\alpha = n + 1$, then $\beta = 1$, and hence

$$\boldsymbol{I}_\alpha^a f(t) = I_a^{n+1}(f(t)) = \frac{1}{\Gamma(n+1)} \int_a^t (t-s)^n f(s) ds$$
$$= \frac{1}{n!} \int_a^t (t-s)^n f(s) ds.$$

It is obvious that if $f \in \boldsymbol{C}(J, \mathbb{R})$, then $\boldsymbol{I}_\alpha^a f(t) \in \boldsymbol{C}(J, \mathbb{R})$, and if $x \in \boldsymbol{C}^{n+1}(J, \mathbb{R})$, then $\boldsymbol{T}_\alpha^a x \in \boldsymbol{C}(J, \mathbb{R}), \alpha \in (n, n+1]$.

Lemma 2.4 [11] Let $\alpha \in (n, n+1]$ and $f \in \boldsymbol{C}^{n+1}(J, \mathbb{R})$, then for all $t > a$ we have

$$\boldsymbol{I}_\alpha^a \boldsymbol{T}_\alpha^a f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (t-a)^k.$$

To obtain the solutions of equations (1.1)-(1.3), let's prove two important results.

Lemma 2.5 The conformable integral $\boldsymbol{I}_\alpha^a$ has the linear property:

$$\boldsymbol{I}_\alpha^a\big(c_1 f(t) + c_2 g(t)\big) = c_1 \boldsymbol{I}_\alpha^a f(t) + c_2 \boldsymbol{I}_\alpha^a g(t), \alpha > 0, c_1, c_2 \in \mathbb{R}, f, g \in \boldsymbol{C}(J, \mathbb{R}).$$

Proof. Let $\beta = \alpha - n$, then

$$\boldsymbol{I}_\alpha^a\big(c_1 f(t) + c_2 g(t)\big) = I_a^{n+1}\Big((t-a)^{\beta-1}\big(c_1 f(t) + c_2 g(t)\big)\Big)$$
$$= \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\beta-1}\big(c_1 f(s) + c_2 g(s)\big) ds$$
$$= c_1 I_a^{n+1}\big((t-a)^{\beta-1} f(t)\big) + c_2 I_a^{n+1}\big((t-a)^{\beta-1} g(t)\big)$$
$$= c_1 \boldsymbol{I}_\alpha^a f(t) + c_2 \boldsymbol{I}_\alpha^a g(t). \quad \blacksquare$$

Lemma 2.6 Let $f \in \boldsymbol{C}(J, \mathbb{R})$, then

1. the conformable integral I_α^a has the particular semigroup properties:

$$\begin{cases} \mathrm{I}_\alpha^a f(t) = \mathrm{I}_1^a \mathrm{I}_{\alpha-1}^a f(t), \alpha \in (1,2], \\ \mathrm{I}_\alpha^a f(t) = \mathrm{I}_2^a \mathrm{I}_{\alpha-2}^a f(t), \alpha \in (2,3]. \end{cases}$$

2. the first and second derivatives of the conformable integral $\boldsymbol{I}_\alpha^a$ are given by

$$\frac{d}{dt} \boldsymbol{I}_\alpha^a f(t) = \boldsymbol{I}_{\alpha-1}^a f(t), \frac{d^2}{dt^2} \boldsymbol{I}_\alpha^a f(t) = \boldsymbol{I}_{\alpha-2}^a f(t), \alpha \in (n, n+1].$$

Proof. (1) If $\alpha \in (1,2]$, then $\beta = \alpha - 1$. By using the semigroup property of the Rieamann-Liouville fractional integral, we have

$$\boldsymbol{I}_{\alpha}^{a} f(t)=I_{a}^{2}\left((t-a)^{\alpha-2} f(t)\right)=\int_{a}^{t}(t-s)(s-a)^{\alpha-2} f(s) d s$$

$$=\int_{a}^{t} \int_{s}^{t}(s-a)^{\alpha-2} f(s) d r d s$$

$$=\int_{a}^{t} \int_{a}^{r}(s-a)^{\alpha-2} f(s)\, d s d r$$

$$=\int_{a}^{t} I_{a}^{1}\left((r-a)^{\alpha-2} f(r)\right) d r=\int_{a}^{t} \boldsymbol{I}_{\alpha-1}^{a} f(r) d r$$

$$=I_{a}^{1}\left(\boldsymbol{I}_{\alpha-1}^{a} f(t)\right)=\boldsymbol{I}_{1}^{a} \boldsymbol{I}_{\alpha-1}^{a} f(t).$$

On the other hand, if $\alpha \in(2,3]$, then $\beta=\alpha-2$, we have

$$\boldsymbol{I}_{\alpha}^{a} f(t)=I_{a}^{3}\left((t-a)^{\alpha-3} f(t)\right)=\frac{1}{2!} \int_{a}^{t}(t-s)^{2}(s-a)^{\alpha-3} f(s) d s$$

$$=\int_{a}^{t} \int_{s}^{t}(t-r)(s-a)^{\alpha-3} f(\mathrm{s}) d r d s$$

$$=\int_{a}^{t}(t-r) \int_{a}^{r}(s-a)^{\alpha-3} f(s)\, d s d r$$

$$=\int_{a}^{t}(t-r) I_{a}^{1}\left((r-a)^{\alpha-3} f(r)\right) d r$$

$$=\int_{a}^{t}(t-r) \boldsymbol{I}_{\alpha-2}^{a} f(r) d r=I_{a}^{2}\left(\boldsymbol{I}_{\alpha-2}^{a} f(t)\right)$$

$$=\boldsymbol{I}_{2}^{a} \boldsymbol{I}_{\alpha-2}^{a} f(t).$$

(2) The first derivative follows by the following:

$$\frac{d}{d t} \boldsymbol{I}_{\alpha}^{a} f(t)=\frac{1}{n!} \frac{d}{d t} \int_{a}^{t}(t-s)^{n}(s-a)^{\beta-1} f(s) d s$$

$$=\frac{1}{(n-1)!} \int_{a}^{t}(t-s)^{n-1}(s-a)^{\beta-1} f(s) d s$$

$$=\boldsymbol{I}_{\alpha-1}^{a} f(t).$$

The second derivative follows by applying the derivative of the previous first derivative. This finishes the proof. ■

Next result we obtain a solution for each linear problem corresponding to nonlinear problem (1.1)-(1.3).

Theorem 2.7 Let $\hat{f}_1 \in \boldsymbol{C}(J, \mathbb{R})$, and $x \in \boldsymbol{C}^1(J, \mathbb{R})$. The conformable linear differential equation

$$\begin{cases} \boldsymbol{T}_{\alpha}^{a} x(t) = \hat{f}_1(t), 0 < \alpha \leq 1, \\ x(a) = x_0, \end{cases} \quad (2.1)$$

has a solution given by

$$x(t) = x_0 + \boldsymbol{I}_{\alpha}^{a} \hat{f}_1(t).$$

Proof. Taking the conformable integral $\boldsymbol{I}_{\alpha}^{a}$ to both sides of equation (2.1), and using Lemma 2.4, we obtain

$$\boldsymbol{I}_{\alpha}^{a}\left(\hat{f}_1(t)\right) = \boldsymbol{I}_{\alpha}^{a} \boldsymbol{T}_{\alpha}^{a}(x(t)) = x(t) - x(a).$$

The condition $x(a) = x_0$ implies the required solution. ■

Theorem 2.8 Let $\hat{f}_2 \in \boldsymbol{C}(J, \mathbb{R})$, and $x \in \boldsymbol{C}^2(J, \mathbb{R})$. The conformable linear sequential differential equation

$$\begin{cases} (\boldsymbol{T}_{\alpha}^{a} + \gamma \boldsymbol{T}_{\alpha-1}^{a}) x(t) = \hat{f}_2(t), 1 < \alpha \leq 2, \gamma \in \mathbb{R}, \\ x(a) = x_0, x'(a) = x_1, \end{cases} \quad (2.2)$$

has a solution given by

$$x(t) = x_0 e^{-\gamma(t-a)} + e^{-\gamma t} \int_a^t e^{\gamma s}\left(x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^{a} \hat{f}_2(s)\right) ds.$$

Proof. Taking the conformable integral $\boldsymbol{I}_{\alpha}^{a}$ to both sides of equation (2.2), and using Lemmas 2.5 and 2.6, we obtain

$$\boldsymbol{I}_{\alpha}^{a} \boldsymbol{T}_{\alpha}^{a} x(t) + \gamma \boldsymbol{I}_{1}^{a} \boldsymbol{I}_{\alpha-1}^{a} \boldsymbol{T}_{\alpha-1}^{a} x(t) = \boldsymbol{I}_{\alpha}^{a} \hat{f}_2(t).$$

Using Lemma 2.4, we have

$$\left(x(t) - x_0 - x_1(t-a)\right) + \gamma \boldsymbol{I}_{1}^{a}(x(t) - x_0) = \boldsymbol{I}_{\alpha}^{a} \hat{f}_2(t). \quad (2.3)$$

If $\gamma = 0$, it is easy to obtain that

$$x(t) = x_0 + x_1(t-a) + \boldsymbol{I}_{\alpha}^{a} \hat{f}_2(t). \quad (2.4)$$

For $\gamma \neq 0$, taking the first derivative of (2.3) and using Lemma 2.6 we obtain

$$x'(t) + \gamma x(t) = x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^{a} \hat{f}_2(t).$$

Let $x(t) = e^{-\gamma t} y(t)$, we get

$$y'(t) = e^{\gamma t}\left(x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^{a} \hat{f}_2(t)\right).$$

Taking the integral from a to t with respect to s, and substituting $y(a) = e^{\gamma a} x_0$, then multiplying by $e^{-\gamma t}$ we get

$$x(t) = x_0 e^{-\gamma(t-a)} + e^{-\gamma t} \int_a^t e^{\gamma s}\left(x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^{a} \hat{f}_2(s)\right) ds.$$

This finishes the proof. ■

Theorem 2.9 Let $\hat{f}_3 \in \boldsymbol{C}(J,\mathbb{R})$, and $x \in \boldsymbol{C}^3(J,\mathbb{R})$. The conformable linear sequential differential equation

$$\begin{cases} \left(\boldsymbol{T}_{\alpha}^{a} + \lambda \boldsymbol{T}_{\alpha-1}^{a} + \frac{\lambda^2}{4}\boldsymbol{T}_{\alpha-2}^{a}\right) x(t) = \hat{f}_3(t), 2 < \alpha \leq 3, \lambda \in \mathbb{R}, \\ x(a) = x_0, x'(a) = x_1, x''(a) = x_2, \end{cases} \tag{2.5}$$

has a solution given by ($\lambda \neq 0$)

$$\begin{aligned} x(t) = {} & x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2}x_0\right)(t-a)e^{-\frac{\lambda}{2}(t-a)} \\ & + \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)\left(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a)e^{-\frac{\lambda}{2}(t-a)} \right. \\ & \left. - \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right) + e^{-\frac{\lambda}{2}t}\int_a^t (t-s)e^{\frac{\lambda}{2}s}\boldsymbol{I}_{\alpha-2}^{a}\hat{f}_3(s)ds. \end{aligned}$$

Proof. Applying the conformable integral $\boldsymbol{I}_{\alpha}^{a}$ to equation (2.5) and using Lemmas 2.5, and 2.6, we have

$$\boldsymbol{I}_{\alpha}^{a}\boldsymbol{T}_{\alpha}^{a}x(t) + \lambda \boldsymbol{I}_{1}^{a}\boldsymbol{I}_{\alpha-1}^{a}\boldsymbol{T}_{\alpha-1}^{a}x(t) + \frac{\lambda^2}{4}\boldsymbol{I}_{2}^{a}\boldsymbol{I}_{\alpha-2}^{a}\boldsymbol{T}_{\alpha-2}^{a}x(t) = \boldsymbol{I}_{\alpha}^{a}\hat{f}_3(t).$$

In view of Lemma 2.4, we have

$$\begin{aligned} & \left(x(t) - x_0 - x_1(t-a) - \frac{1}{2}x_2(t-a)^2\right) \\ & \quad + \lambda \boldsymbol{I}_{1}^{a}\big(x(t) - x_0 - x_1(t-a)\big) + \frac{\lambda^2}{4}\boldsymbol{I}_{2}^{a}(x(t) - x_0) \\ & \quad = \boldsymbol{I}_{\alpha}^{a}\hat{f}_3(t). \end{aligned} \tag{2.6}$$

If $\lambda = 0$, then it is obvious that

$$\begin{aligned} x(t) = {} & x_0 + x_1(t-a) + \frac{1}{2}x_2(t-a)^2 \\ & + \boldsymbol{I}_{\alpha}^{a}\hat{f}_3 \end{aligned} \tag{2.7}$$

For $\lambda \neq 0$, taking the first and second derivatives of (2.6) and using Lemma 2.6, we have

$$\begin{aligned} & x'(t) - x_1 - x_2(t-a) + \lambda\big(x(t) - x_0 - x_1(t-a)\big) \\ & \quad + \frac{\lambda^2}{4}\boldsymbol{I}_{1}^{a}(x(t) - x_0) = \boldsymbol{I}_{\alpha-1}^{a}\hat{f}_3(t), \end{aligned}$$

and

$$x''(t) + \lambda x'(t) + \frac{\lambda^2}{4}x(t) = \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right) + \boldsymbol{I}_{\alpha-2}^{a}\hat{f}_3(t).$$

Let $x(t) = e^{\frac{-\lambda}{2}t}y(t)$, then

$$y''(t) = \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)e^{\frac{\lambda}{2}t} + e^{\frac{\lambda}{2}t}\boldsymbol{I}_{\alpha-2}^{a}\hat{f}_3(t).$$

Taking the double integral and then multiplying by $e^{-\frac{\lambda}{2}t}$, we get

$$x(t) = x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2}x_0\right)(t-a)e^{-\frac{\lambda}{2}(t-a)} + \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)\left(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a)e^{-\frac{\lambda}{2}(t-a)} - \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right) + e^{-\frac{\lambda}{2}t}\int_a^t (t-s)e^{\frac{\lambda}{2}s}\boldsymbol{I}_{\alpha-2}^{a}\hat{f}_3(s)ds.$$

This finishes the proof. ■

Remark 2.10 The classical integer order differential equations can be obtained in the following sense:

1. When $\alpha \to 1^-$ in Theorem 2.7, the solution of (2.1) is given by

$$x(t) = x_0 + \int_a^t \hat{f}_1(s)ds.$$

2. When $\alpha \to 2^-$ in Theorem 2.8, the solution of (2.2) is given by

$$x(t) = x_0 + \frac{x_1}{\gamma}\left(1 - e^{-\gamma(t-a)}\right) + \frac{e^{-\gamma t}}{\gamma}\int_a^t (e^{\gamma r} - e^{\gamma a})(r-a)^{\alpha-2}\hat{f}_2(r)dr.$$

3. When $\alpha \to 3^-$ in Theorem 2.9, the solution of (2.5) is given by

$$x(t) = x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2}x_0\right)(t-a)e^{-\frac{\lambda}{2}(t-a)} + \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)\left(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a)e^{-\frac{\lambda}{2}(t-a)} - \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right) + e^{-\frac{\lambda}{2}t}\int_a^t\int_a^s (t-s)e^{\frac{\lambda}{2}s}(r-a)^{\alpha-3}\hat{f}_3(s)dr\,ds.$$

We close this section by the so-called Schauder fixed point theorem.

Theorem 2.11[14] If F is a closed , bounded, convex subset of a Banach space X and the mapping $\Delta: U \to U$ is completely continuous, then Δ has a fixed point in F.

3 Existence Theorems

The fixed point theorems are the basic tools for dealing with the nonlinear differential equations (1.1)-(1.3). The idea is to convert the corresponding integral equation into operator equation and then proving this equation has a fixed point, which is then the required solution. We shall focus on two fixed point theorems, the Banach and Schauder fixed point theorems [14]. We assume hereafter that $\gamma \neq 0$ (see equation 2.4) and $\lambda \neq 0$ (see equation 2.7), otherwise the next results will be similar to the case when $\alpha \in (0,1]$.

In view of Theorems 2.7-2.9 and equations (1.1)-(1.3), we define the operators Ψ_1, Ψ_2 and Ψ_3 on $\boldsymbol{C}(J, \mathbb{R})$, as

$$\Psi_1 x(t) = x_0 + \boldsymbol{I}_\alpha^a f_1(t, x(t)),$$

$$\Psi_2 x(t) = x_0 e^{-\gamma(t-a)} + e^{-\gamma t} \int_a^t e^{\gamma s}\left(x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^a f_2(s, x(s))\right) ds,$$

and

$$\begin{aligned}\Psi_3 x(t) = {} & x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2} x_0\right)(t-a) e^{-\frac{\lambda}{2}(t-a)} \\ & + \left(x_2 + \lambda x_1 + \frac{\lambda^2}{4} x_0\right)\left(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a) e^{-\frac{\lambda}{2}(t-a)} \right. \\ & \left. - \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right) \\ & + e^{-\frac{\lambda}{2}t} \int_a^t (t-s) e^{\frac{\lambda}{2}s} \boldsymbol{I}_{\alpha-2}^a f_3(s, x(s)) ds.\end{aligned}$$

Theorem 3.1. The operators $\Psi_k, k = 1,2,3,$ are completely continuous.

Proof. The continuity of the operators $\Psi_k, k = 1,2,3,$ follows respectively by the continuity of the functions f_1, f_2, and f_3. Let $\mathcal{B}$ be a bounded proper subset of $\boldsymbol{C}(J, \mathbb{R})$, then, there exist positive real numbers A_k, $k = 1,2,3,$ such that $|f_k(t, x)| \leq A_k$, for any order pair $(t, x) \in J \times \mathcal{B}$. Therefore

$$|\Psi_1 x(t)| \le |x_0| + |\boldsymbol{I}_\alpha^a f_1(t,x(t))|$$
$$\le |x_0| + \left|\int_a^t (s-a)^{\alpha-1} f_1(s,x(s))ds\right|$$
$$\le |x_0| + \frac{A_1(t-a)^\alpha}{\alpha},$$

$$|\Psi_2 x(t)| \le |x_0|e^{-\gamma(t-a)}$$
$$+ e^{-\gamma t}\int_a^t e^{\gamma s}\left|x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^a f_2(s,x(s))\right|ds$$
$$\le |x_0|e^{-\gamma(t-a)}$$
$$+ e^{-\gamma t}\int_a^t e^{\gamma s}\Bigg(|x_1| + \gamma|x_0|$$
$$+ \left|\int_a^s (u-a)^{\alpha-2} f_2(u,x(u))du\right|\Bigg)ds$$
$$\le |x_0|e^{-\gamma(t-a)} + \frac{\left|1-e^{-\gamma(t-a)}\right|}{\gamma}\left(|x_1| + |\gamma x_0| + \frac{A_2(t-a)^{\alpha-1}}{\alpha-1}\right),$$

and

$$|\Psi_3 x(t)| \le \left|x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2}x_0\right)(t-a)e^{-\frac{\lambda}{2}(t-a)}\right|$$
$$+ \Bigg|\left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)\Bigg(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a)e^{-\frac{\lambda}{2}(t-a)}$$
$$- \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\Bigg)\Bigg|$$
$$+ \left|e^{-\frac{\lambda}{2}t}\int_a^t (t-s)e^{\frac{\lambda}{2}s}\boldsymbol{I}_{\alpha-2}^a f_3(s,x(s))ds\right|$$
$$\le \left|x_0 e^{-\frac{\lambda}{2}(t-a)} + \left(x_1 + \frac{\lambda}{2}x_0\right)(t-a)e^{-\frac{\lambda}{2}(t-a)}\right|$$
$$+ \left|\left(x_2 + \lambda x_1 + \frac{\lambda^2}{4}x_0\right)\left(\left(\frac{2}{\lambda}\right)^2 - \frac{2}{\lambda}(t-a)e^{-\frac{\lambda}{2}(t-a)} - \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right)\right|$$
$$+ \left|e^{-\frac{\lambda}{2}t}\int_a^t (t-s)e^{\frac{\lambda}{2}s}\left(\int_a^s (u-a)^{\alpha-3} f_3(u,x(u))du\right)ds\right|.$$

The inequalities $(t-s) \le (t-a)$, and $(s-a)^{\alpha-2} \le (t-a)^{\alpha-2}$, imply that

$$|\Psi_3 x(t)| \le |x_0| e^{-\frac{\lambda}{2}(t-a)} + \left(|x_1| + \frac{|\lambda x_0|}{2}\right)(t-a)e^{-\frac{\lambda}{2}(t-a)}$$
$$+\left(|x_2| + |\lambda x_1| + \frac{\lambda^2}{4}|x_0|\right)\left(\left(\frac{2}{\lambda}\right)^2 + \frac{2}{|\lambda|}(t-a)e^{-\frac{\lambda}{2}(t-a)}\right.$$
$$\left. + \left(\frac{2}{\lambda}\right)^2 e^{-\frac{\lambda}{2}(t-a)}\right)$$
$$+\frac{2A_3}{(\alpha-2)|\lambda|}\left|1 - e^{-\frac{\lambda}{2}(t-a)}\right|(t-a)^{\alpha-1}.$$

Taking the maximum over J, we deduce that the operators $\Psi_k, k = 1,2,3$, are uniformly bounded on $\mathcal{B}$. Next, we show the equicontinuity of $\Psi_k, k = 1,2,3$. For this, let $a \le t_1 < t_2 \le T$, then

$$|\Psi_1 x(t_2) - \Psi_1 x(t_1)| \le \int_{t_1}^{t_2} (s-a)^{\alpha-1} |f_1(s, x(s))| ds,$$

$$|\Psi_2 x(t_2) - \Psi_2 x(t_1)|$$
$$\le |x_0|\left|1 - e^{\gamma(t_2-t_1)}\right| e^{-\gamma(t_2-a)}$$
$$+e^{-\gamma(t_2-a)}\left|1 - e^{\gamma(t_2-t_1)}\right| \int_a^{t_1} e^{\gamma s}\left|x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^a f_2\big(s, x(s)\big)\right| ds$$
$$+e^{\gamma t_2} \int_{t_1}^{t_2} e^{\gamma s}\left|x_1 + \gamma x_0 + \boldsymbol{I}_{\alpha-1}^a f_2\big(s, x(s)\big)\right| ds,$$

and

$$|(\Psi_3 x)(t_2) - (\Psi_3 x)(t_1)|$$
$$\le |x_0| e^{-\frac{\lambda}{2}(t_2-a)}\left|1 - e^{\frac{\lambda}{2}(t_2-t_1)}\right|$$
$$+\left(|x_1| + \frac{|\lambda x_0|}{2}\right)\left(a e^{-\frac{\lambda}{2}(t_2-a)}\left|e^{\frac{\lambda}{2}(t_2-t_1)} - 1\right|\right.$$
$$\left. + \left(t_2 e^{-\frac{\lambda}{2}(t_2-a)}\left|1 - e^{\frac{\lambda}{2}(t_2-t_1)}\right| + (t_2 - t_1)e^{-\frac{\lambda}{2}(t_1-a)}\right)\right)$$
$$+\left(|x_2| + |\lambda x_1| + \frac{\lambda^2}{4}|x_0|\right)$$

$$\times\left(\frac{2}{|\lambda|}\left(ae^{-\frac{\lambda}{2}(t_2-a)}\left|e^{\frac{\lambda}{2}(t_2-t_1)}-1\right|\right.\right.$$
$$\left.+\left(t_2e^{-\frac{\lambda}{2}(t_2-a)}\left|1-e^{\frac{\lambda}{2}(t_2-t_1)}\right|+(t_2-t_1)e^{-\frac{\lambda}{2}(t_1-a)}\right)\right)$$
$$\left.+\left(\frac{2}{\lambda}\right)^2e^{-\frac{\lambda}{2}(t_2-a)}\left|1-e^{\frac{\lambda}{2}(t_2-t_1)}\right|\right)$$
$$+\left|(t_2-a)e^{-\frac{\lambda}{2}t_2}\left(1-e^{\frac{\lambda}{2}(t_2-t_1)}\right)\right.$$
$$\left.+(t_2-t_1)e^{-\frac{\lambda}{2}t_1}\right|\int_a^{t_1}e^{\frac{\lambda}{2}s}\boldsymbol{I}_{\alpha-2}^{a}\left|f_3\big(s,x(s)\big)\right|ds$$
$$+e^{-\frac{\lambda}{2}t_2}\int_{t_1}^{t_2}(t_2-s)e^{\frac{\lambda}{2}s}\boldsymbol{I}_{\alpha-2}^{a}|f_3(s,x(s))|ds.$$

Using dominated convergence theorem and as $|t_2-t_1|\to 0$, then

$$|\Psi_kx(t_2)-\Psi_kx(t_1)|\to 0, k=1,2,3.$$

We conclude that Ψ_k, $k=1,2,3$ are all equicontinuous on J. In consequence, it follows by the Arzela-Ascoli theorem that the operators Ψ_k, $k=1,2,3$ are completely continuous. This finishes the proof.■

According to Theorem 2.11, if we define a closed, bounded, convex subset F of $\boldsymbol{C}(J,\mathbb{R})$ on which the operators Ψ_k, $k=1,2,3$, are completely continuous, then the problems (1.1)-(1.3) have the respective solution.

Theorem 3.3 Let B_k, *k=1,2,3,* be positive constants such that

$$\lim_{x\to 0}\frac{f_k(t,x)}{x}=\mu_k(t)\leq B_k, k=1,2,3,$$

then, each problem of (1.1)-(1.3) has a solution.

Proof. The given conditions imply that there exist positive constants $C_k, k=1,2,3,$ such that

$$\left|f_k\big(t,x(t)\big)\right|\leq(1+\mu_k(t))|x|\leq(1+B_k)C_k.$$

Define the subsets F_k, of $\boldsymbol{C}(J,\mathbb{R})$ as

$$F_k=\{x\in\boldsymbol{C}(J,\mathbb{R}):|x(t)|\leq C_k, t\in J\},$$

for $k=1,2,3$. Hence all F_k are closed, bounded, and convex subsets of $\boldsymbol{C}(J,\mathbb{R})$. By Theorem 3.1, the operators Ψ_k are completely continuous, then by Schauder fixed point Theorem 2.11, each problem of (1.1)-(1.3) has a solution. This finishes the proof.■

Next result, we show the existence and uniqueness of solution for each problem of (1.1)-(1.3) by using the contraction principle and the so-called Banach fixed point theorem.

Theorem 3.4. Let $f_k, k = 1,2,3$ be Lipschitz functions that are satisfying the conditions

$$|f_k(t,x) - f_k(t,y)| \le C_k|x-y|,$$

where $t \in J, x, y \in \mathbb{R}$, and $C_k > 0$. Then, each problem of (1.1)-(1.3) has a unique solution whenever

$$\tau_1 = \frac{C_1(T-a)^\alpha}{\alpha} < 1,$$

$$\tau_2 = \frac{C_2(T-a)^{\alpha-1}}{|\gamma|(\alpha-1)}\left(1 + e^{|\gamma|(T-a)}\right) < 1,$$

and

$$\tau_3 = \frac{2C_3}{(\alpha-2)|\lambda|}\left(1 + e^{\frac{|\lambda|}{2}(T-a)}\right)(T-a)^{\alpha-1} < 1.$$

Proof. The continuity of f_k implies that there exists positive constant D_k such that $\max\{|f_k(t,0)|: t \in J\} \le D_k$ for each $k = 1,2,3$. We show firstly that $\Psi_k \mathfrak{B}_{r_k} \subset \mathfrak{B}_{r_k}$, where $\mathfrak{B}_{r_k}$is defined by $\mathfrak{B}_{r_k} = \{x \in \boldsymbol{C}(J,\mathbb{R}) : \|x\| \le r_k\}$, and $r_k, k = 1,2,3$, are given by

$$r_2 > (1-\tau_2)^{-1}\left(|x_0| + \frac{D_2(T-a)^\alpha}{\alpha}\right)$$

$$r_2 > (1-\tau_2)^{-1}\left(|x_0|e^{|\gamma|(T-a)} + \frac{1}{|\gamma|}\left(|x_1| + |\gamma x_0| + \frac{D_2(T-a)^{\alpha-1}}{\alpha-1}\right)(1 +\right.$$

eγT−a,

$$r_3 > (1-\tau_3)^{-1}\left(|x_0|e^{\frac{|\lambda|}{2}(T-a)} + \left(|x_1| + \frac{|\lambda x_0|}{2}\right)(T-a)e^{\frac{|\lambda|}{2}(T-a)} +\right.$$

x2+ λx1+ λ24x0

2λ2+2λT−aeλ2T−a+2λ2eλ2T−a+2D3α−2λ1+ eλ2T−aT−aα−1

For doing this, let $x \in \mathfrak{B}_{r_1}$ then

$$|\Psi_1 x(t)| \le |x_0| + \frac{(C_1\|x\| + D_1)(T-a)^\alpha}{\alpha} \le (1-\tau_1)r_1 + \tau_1 r_1 = r_1.$$

For $x \in \mathfrak{B}_{r_2}$, we have

$$|\Psi_2 x(t)| \le |x_0| e^{|\gamma|(T-a)} + \frac{1}{|\gamma|}\left(|x_1| + |\gamma x_0| + \frac{(C_2\|x\| + D_2)(T-a)^{\alpha-1}}{(\alpha-1)}\right)\left(1 + e^{|\gamma|(T-a)}\right) \le (1-\tau_2)r_2 + \tau_2 r_2 = r_2.$$

For $x \in \mathfrak{B}_{r_3}$, we have

$$|\Psi_3 x(t)| \le |x_0| e^{\frac{|\lambda|}{2}(T-a)} + \left(|x_1| + \frac{|\lambda x_0|}{2}\right)(T-a)e^{\frac{|\lambda|}{2}(T-a)} + \left(|x_2| + |\lambda x_1| + \frac{\lambda^2}{4}|x_0|\right)\left(\left(\frac{2}{\lambda}\right)^2 + \frac{2}{|\lambda|}(T-a)e^{\frac{|\lambda|}{2}(T-a)} + \left(\frac{2}{\lambda}\right)^2 e^{\frac{|\lambda|}{2}(T-a)}\right) + \frac{2(C_3\|x\| + D_3)}{(\alpha-2)|\lambda|}\left(1 + e^{\frac{|\lambda|}{2}(T-a)}\right)(T-a)^{\alpha-1} \le (1-\tau_3)r_3 + \tau_3 r_3 = r_3.$$

Next step is showing the contraction principle. For doing this, let $x, y \in \boldsymbol{C}(J, \mathbb{R})$,then

$$|\Psi_1 x(t) - \Psi_1 y(t)| \le \frac{C_1(T-a)^{\alpha}}{\alpha}\|x-y\|,$$

$$|\Psi_2 x(t) - \Psi_2 y(t)| \le \frac{C_2(T-a)^{\alpha-1}}{|\gamma|(\alpha-1)}\left(1 + e^{|\gamma|(T-a)}\right)\|x-y\|,$$

$$|\Psi_3 x(t) - \Psi_3 y(t)| \le \frac{2C_3(T-a)^{\alpha-1}}{|\lambda|(\alpha-2)}\left(1 + e^{\frac{|\lambda|}{2}(T-a)}\right)\|x-y\|,$$

As $\tau_k < 1, k = 1,2,3,$ the contraction principles are satisfied. By Banach fixed point theorem, there exists a unique solution for each problem of (1.1)-(1.3). The proof is completed.■

We close this article by the following examples.

Example 3.5 Consider the following fractional sequential differential equation

$$\begin{cases} (\boldsymbol{T}_{1.5}^{0} - \boldsymbol{T}_{0.5}^{0})x(t) = Ct \sin x(t), t \in (0,1], \\ x(0) = 0, x'(0) = \frac{\pi}{2}. \end{cases} \tag{3.1}$$

Here, $\alpha = 1.5$, $\gamma = -1$, and $f_2(t, x(t)) = Ct \sin x(t) \le Ct$. We notice that

$|f_2(t,x) - f_2(t,y)| \leq Ct|x-y| < C|x-y|$, $\lim_{x\to 0} \frac{f_2(t,x)}{x} = Ct \leq C$, for any $t\in 0,1$, and $\tau 2 = C0.51+e \approx 7.44C$. Therefore, Theorems 3.3, and 3.4 can be applied if we choose $0 < C \leq 0.134$, then the problem (3.1) has a unique solution in $\boldsymbol{C}([0,1],\mathbb{R})$.

Example 3.6 Consider the following nonlinear fractional initial value problem:

$$\begin{cases} (\boldsymbol{T}_{2.5}^2 + 2\boldsymbol{T}_{1.5}^2 + \boldsymbol{T}_{0.5}^2)x(t) = D\left(\dfrac{|x(t)|}{1+t|x(t)|}\right), t \in (1,2], \\ \\ x^{(k)}(1) = 1, \ \ k = 0,1,2. \end{cases} \tag{3.2}$$

Here $f_3(t,x(t)) = D\left(\frac{|x(t)|}{1+t|x(t)|}\right) \leq \frac{|D|}{t} \leq |D|$, $\alpha = 2.5$, $\lambda = 2$. Then

$$|f(t,x) - f(t,y)| \ \leq D|x-y|, \forall\, t \in [1,2],$$

and $\tau_3 = \frac{D}{0.5}(1+e) \approx 7.44D$. Therefore, choosing $0 < D \leq 0.134$, the Theorems 3.3 and 3.4 can be applied, hence the problem (3.2) has a unique solution in $\boldsymbol{C}([1,2],\mathbb{R})$.

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